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A higher-order asymptotic solution for heat transfer of a laminar flow passing a wedge at small Prandtl number

YE-MON CHEN
Department of Chemical Engineering and Technology, National Taiwan Institute of Technology, Taipei, Taiwan 107, Republic of China

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1. INTRODUCTION

THE PRESENT paper is a continuation of a previous work [1] to find a better asymptote for heat transfer at small Prandtl number. The conventional approach is to assume that the velocity boundary layer is negligibly thin and a uniform velocity field, shown in Fig. 1a, is applied in the entire region of the energy equation. Obviously, the uniform velocity field leads to an unrealistic slip condition on the wall, which results in a non-vanishing convective term, and the heat transfer rate is always overestimated. In the previous approach, called the two-region model, it was found that within the velocity boundary layer, the convective heat transfer terms are negligible compared to conductive term and the velocity field is displaced into a step change, shown in Fig. 1b. Within the velocity boundary layer, the velocity field vanishes, and becomes uniform beyond the layer. Although the two-region model yields a much better prediction, it also leads to an underestimation, resulting from neglection of the convective terms.

Since the conventional asymptote always overestimates and the two-region model always underestimates, any reasonable assumption for the velocity field that lies between these two models should result in a heat transfer rate that falls between over- and underestimation, and, hopefully, this may lead to a better prediction. To make an explicit solution possible, a linear velocity profile within the velocity boundary layer is chosen, as shown in Fig. 1c. On the other hand, since the present theory lies between the two extremes, there is no guarantee as to whether it will over- or underestimate.

2. ANALYSIS

When the fluid properties are considered uniform and the heat generation due to viscous dissipation is negligible [1],

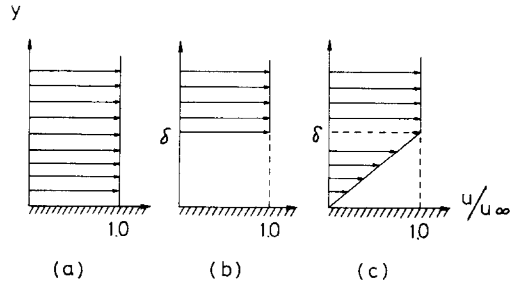

FIG. 1. Velocity fields of (a) the conventional asymptote; (b) the two-region model [1]; and (c) the present model.

the energy equation is given by

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \quad (1)$$

For the case of uniform wall temperature, the boundary conditions are

$$y=0, \quad T=T_w \quad (2)$$

$$y\to\infty, \quad T=T_\infty. \quad (3)$$

The velocity component in the x-direction, u, is assumed to be linear within the velocity boundary layer and uniform

NOMENCLATURE

a	constant defined by equation (9)
c	constant defined by equation (8) [$m^{1-m}\,s^{-1}$]
h	heat transfer coefficient [$J\,m^{-2}\,s^{-1}\,K^{-1}$]
k	thermal conductivity [$J\,m^{-1}\,s^{-1}\,K^{-1}$]
m	exponent defined by equation (8)
Nu	Nusselt number
Pr	Prandtl number
q	heat flux in the y-direction [$J\,m^{-2}\,s^{-1}$]
Re	Reynolds number
T	temperature [K]
u	velocity component in the x-direction [$m\,s^{-1}$]
v	velocity component in the y-direction [$m\,s^{-1}$]
x	coordinate along the wedge [m]
y	coordinate normal to the wedge [m].

Greek symbols

α	thermal diffusivity [$m^2\,s^{-1}$]
β	angle factor of the wedge
η	dimensionless variable defined by equation (14)
ν	kinetic viscosity [$m^2\,s^{-1}$]
Γ	gamma function
δ	displacement boundary-layer thickness [m]
ζ	dimensionless variable defined by equation (12).

Subscripts

w	condition on the wall
x	local value at position x
∞	condition in the bulk flow
δ	condition at the position of δ.

Superscript

*	pseudo-boundary condition.

beyond the layer,

$$u = u_\infty(x)\frac{y}{\delta(x)}, \quad 0 \leqslant y \leqslant \delta \tag{4}$$

$$u = u_\infty(x), \quad y > \delta. \tag{5}$$

The velocity component in the y-direction, v, can be obtained by integrating the continuity equation,

$$v = -\frac{y^2}{2}\frac{d}{dx}\left(\frac{u_\infty}{\delta}\right) \quad 0 \leqslant y \leqslant \delta \tag{6}$$

$$v = -y\frac{du_\infty}{dx} \quad y > \delta. \tag{7}$$

For a wedge of angle $\beta\pi$, the potential flow is given by

$$u_\infty(x) = c\cdot x^m = c\cdot x^{\beta/2-\beta} \tag{8}$$

and δ was defined [1] as the displacement thickness,

$$\delta(x) = a\cdot x\cdot Re_x^{-1/2}. \tag{9}$$

Values of the constant a for various wedge angles were listed in ref. [1].

The solutions of the energy equation within the velocity boundary layer and beyond the layer are obtained separately by applying different velocity fields of equations (4)–(7) in different regions. With velocity field of equations (5) and (7), boundary condition (3) and a pseudo-boundary condition

$$y = \delta, \quad T = T_\delta^* \tag{10}$$

the energy equation beyond the layer is integrated [1] to

$$\frac{T-T_\infty}{T_\delta^*-T_\infty} = 1 - \frac{2}{\sqrt{\pi}}\int_0^\zeta \exp(-\zeta^2)\,d\zeta \tag{11}$$

where

$$\zeta = \frac{y-\delta}{2x}(m+1)^{1/2}(Re_x Pr)^{1/2}. \tag{12}$$

With equations (1), (4) and (6), the energy equation within the velocity boundary layer becomes

$$u_\infty\frac{y}{\delta}\frac{\partial T}{\partial x} - \frac{y^2}{2}\frac{d}{dx}\left(\frac{u_\infty}{\delta}\right)\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{13}$$

By introducing another combined variable

$$\eta = y\left[\frac{(m+1)u_\infty}{12\alpha\delta x}\right]^{1/3} = \frac{y}{x}\left[\frac{(m+1)}{12a}Re_x^{3/2}\,Pr\right]^{1/3} \tag{14}$$

equation (13) is transformed to

$$\frac{d^2T}{d\eta^2} + 3\eta^2\frac{dT}{d\eta} = 0. \tag{15}$$

With another pseudo-boundary condition

$$y = \infty, \quad T = T_\infty^* \tag{16}$$

and the boundary condition of equation (2), equation (15) can be integrated to

$$\frac{T-T_\infty^*}{T_w-T_\infty^*} = 1 - \frac{1}{\Gamma(4/3)}\int_0^\eta \exp(-\eta^3)\,d\eta. \tag{17}$$

The two pseudo-boundary conditions, equations (10) and (16), can be eliminated by matching the inner and outer solutions, (11) and (17). The matching process requires that the temperature and the heat flux be continuous at the boundary, $y = \delta$.

To have a continuous temperature, it is necessary that

$$\frac{T_\delta^*-T_\infty^*}{T_w-T_\infty^*} = 1 - \frac{1}{\Gamma(4/3)}\int_0^{\eta_\delta}\exp(-\eta^3)\,d\eta \tag{18}$$

where

$$\eta_\delta = \left[\frac{a^2}{12}(m+1)Pr\right]^{1/3}.$$

To have a continuous heat flux, from equation (11),

$$q|_{y=\delta} = \frac{k}{x}(T_\delta^*-T_\infty)\left(\frac{m+1}{\pi}\right)^{1/2}(Re_x\,Pr)^{1/2} \tag{19}$$

from equation (17),

$$q|_{y=\delta} = \frac{k}{x}(T_w-T_\infty^*) \times\frac{\exp(-\eta_\delta^3)}{\Gamma(4/3)}\left[\frac{(m+1)}{12a}Re_x^{3/2}\,Pr\right]^{1/3} \tag{20}$$

it is necessary that equations (19) and (20) be equal,

$$\frac{T_\delta^*-T_\infty}{T_w-T_\infty^*} = \frac{\exp(-\eta_\delta^3)}{\Gamma(4/3)\pi^{-1/2}[(12a)^2(m+1)Pr]^{1/6}}. \tag{21}$$

Heat flux on the wall is given by

$$q|_{y=0} = \frac{k}{x}(T_w-T_\infty^*)\frac{1}{\Gamma(4/3)}\left[\frac{(m+1)}{12a}Re_x^{3/2}\,Pr\right]^{1/3} = h_x(T_w-T_\infty). \tag{22}$$

Combine equations (18), (21) and (22),

$$Nu_x = \frac{(m+1)^{1/2}(Re_x\,Pr/\pi)^{1/2}}{\pi^{-1/2}[(12a)^2(m+1)\,Pr]^{1/6}\int_0^{\eta_\delta}\exp(-\eta^3)\,d\eta + \exp(-\eta_\delta^3)}. \tag{23}$$

3. SOLUTIONS AT EXTREME VALUES OF PRANDTL NUMBER

As the Prandtl number approaches zero, the first term of the denominator in equation (23) vanishes and the second term approaches unity. Therefore,

$$Nu_x = (m+1)^{1/2}(Pr\,Re_x/\pi)^{1/2}, \quad Pr\to 0 \tag{24}$$

which reduces to the conventional asymptote. This indicates that equation (23) is also an asymptotic solution for small Prandtl numbers.

On the other hand, as the Prandtl number approaches infinity, the second term in the denominator vanishes and the integration term approaches $\Gamma(4/3)$. Therefore,

$$Nu_x = \left(\frac{m+1}{12a}\right)^{1/3}\frac{1}{\Gamma(4/3)}Re_x^{1/2}\,Pr^{1/3}, \quad Pr\to\infty. \tag{25}$$

Equation (25) yields a correct dependence of 1/3 power on the Prandtl number, though the proportional constant is inaccurate. For the example of a flat plate, the proportional constant given by (25) is 0.408 while the exact solution is 0.339.

4. RESULTS AND DISCUSSION

Calculations of equation (23) for the special case of a flat plate, $\beta = 0$, for a wide range of Prandtl numbers, 0.001–100, are compared with the numerical data of the exact solution [2] in Fig. 2. In this figure, equation (23) is represented by a solid line and the exact solution by open circles connected with a broken line. It is seen that the two lines coincide up to $Pr = 1$,

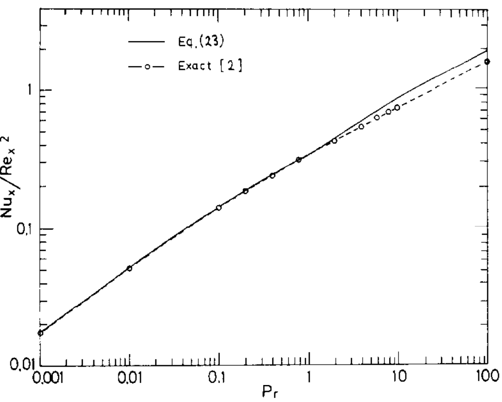

FIG. 2. Comparison of the exact solution and equation (23) for the case of a flat plate, $\beta = 0$.

start to diverge for larger Prandtl numbers and, finally, become parallel to each other at a sufficiently large Prandtl number. For $Pr \leqslant 1$, the maximum error of 70% by the conventional asymptote was reduced to 13% by the two-region model [1] and it is further reduced to only 0.8% by the present theory. The accuracy is dramatically improved, particularly in the range of $0.1 \leqslant Pr \leqslant 1$. It should also be noticed that the present theory converges faster than the other two asymptotes at a small Prandtl number.

Calculations also indicate that the present theory shifts from underestimation to overestimation between Prandtl numbers of 0.8 and 1.0, which implies that there is one point between these two values at which the present theory and the exact solution yield identical values. Since equation (23) is an asymptote for small Prandtl numbers, as indicated by (24), it also matches the exact solution asymptotically at zero Prandtl number. Therefore, the present theory matches the exact solution at two points. On the other hand, the two-region model and conventional asymptote both match the exact solution asymptotically at zero Prandtl number, but deviate ever afterwards, with the two-region model always underestimating and the convectional asymptote overestimating. In the sense of the number of points matching the exact solution, the two-region model and the conventional asymptote are of first-order, and the present theory is a second-order asymptote.

For an accelerating flow, $\beta > 0$, the present theory also yields better predictions and converges faster than the other two asymptotes. For instance, the maximum error for the case of $\beta = 2$ for $Pr \leqslant 1$ is 2.5%, compared to 5.6% by the two-region model [1] and 32% by the conventional asymptote. For a decelerating flow when the diverging angle is not too large, the present theory again gives better predictions. For $Pr \leqslant 1$, the maximum error is 3.3% for $\beta = -0.1$, compared to 15% by the two-region model, and 8.4% for $\beta = -0.16$, compared to 16% [1]. However, for a strong decelerating flow, $\beta = -0.198838$, the present theory yields results with about the same accuracy as the two-region model up to $Pr = 0.1$, but becomes less accurate for larger Prandtl numbers. At $Pr = 1$, the error is 26% compared to 15% for the previous model.

It is interesting to see that the present theory has its best prediction for the case of a flat plate. This can be explained by the following argument.

Since the heat transfer rate depends strongly on the velocity field close to the wall, a good velocity simulation in this region is essential to the heat transfer rate prediction. Under the non-slip and impermeable conditions, the velocity boundary-layer equation on the wall becomes

$$\nu \left(\frac{\partial^2 u}{\partial y^2} \right) \bigg|_{y=0} = -u_\infty \frac{du_\infty}{dx}. \tag{26}$$

For the special case of a flat plate, $du_\infty/dx = 0$, the second derivative is identical to zero. This indicates that the velocity profile close to the wall can be well approximated by a linear profile. Incidentally, as a linear profile is assumed in the velocity boundary layer by the present theory, there is good reason to believe that the present theory is at its best for the case of a flat plate.

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Double diffusion from a horizontal line source in an infinite porous medium

STEVEN E. LARSON† and DIMOS POULIKAKOS‡

† Engine Electronics Automotive and Industrial Electronics Group, Motorola, Inc., 1299 E. Algonquin Road, Schaumburg, IL 60196, U.S.A.

‡ Department of Mechanical Engineering, University of Illinois at Chicago, P.O. Box 4348, Chicago, IL 60680, U.S.A.

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1. INTRODUCTION

THE PRESENT paper aims to analyze an important fundamental problem in porous media natural convection: the phenomenon of time-dependent heat, mass and fluid flow induced by a horizontal line source producing simultaneously heat and a chemical species. The study will determine the effect of the presence of the chemical species on the main features of the flow field which, after it originates in the vicinity of the line source, penetrates the unbounded porous surroundings. In addition to its fundamental nature, the present problem finds practical applications illustrated by the spreading of chemical pollutants generated by exothermic reactions in the earth's crust, the chemical industry, the disposal of nuclear wastes and the natural convection cooling of buried electrical cables.

Previous studies of natural convection from a line source